First- and Second-Order Sensitivity Matrices (Differential Models) of Electric Power Systems: Applicability for Post-Emergency Steady-State Analysis

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***Abstract*— Reliability assessment of electric power systems necessitates an analysis of numerous operating states involving potential equipment outages and stochastic events within the minimal time. The sensitivity matrix method enables the calculation of post-emergency steady states. This paper describes the first- and second-order differential models for electric power system steady-state analysis, representing the dependence of bus voltage variations on power fluctuations. These models are applied to simulate generator outages and consumer load fluctuations. The proposed differential models are validated on a test electric power system. The first- and second-order sensitivity matrices applied to the model of electric power system steady states in power balance form demonstrate accurate qualitative and quantitative approximation of operating parameters. Furthermore, the second-order sensitivity matrices markedly improve the calculation accuracy of post-emergency steady-states.**

***Index Terms* — Electrical power system, operational reliability, post-emergency steady state, differential models, sensitivity matrices.**

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I. Introduction

Electric power systems (EPSs) function across a range of operating states. Transitions between them are driven by specific events. Figure 1 illustrates the classification of the EPS states and their transitions [1, 2].

Operational reliability is fully ensured in the normal steady state of the EPS, specifically, when the reliability indices exceed the standard (critical) thresholds. Under normal steady states, the EPS components fully satisfy the demand for electricity and power while meeting power quality standards. Operational parameters remain within permissible limits in accordance with techno-economic characteristics of the consumers.

Due to specific events, an EPS may transition from a normal steady state to a heavy-load steady state, where almost all operational parameters are at the boundary of permissible limits without violating them. Even though all reserve margins are exhausted, the load remains fully supplied. Transitioning the EPS to a normal operating state and enhancing its reliability require a set of control actions.

Adverse events may trigger a transition of the EPS from normal or heavy-load steady state to an emergency state. In an emergency state, almost all operational parameters violate the permissible limits, all types of available reserves are exhausted, and a portion of the electricity and power consumers is disconnected. Restoring the EPS to more reliable states requires a series of control actions, such as load shedding, cold reserve activation, and the rapid repair or replacement of faulty components.

The emergency state and system failure are followed by the post-emergency steady state, where certain EPS components fail or are taken out of service for repair due to an emergency outage. The remaining components may

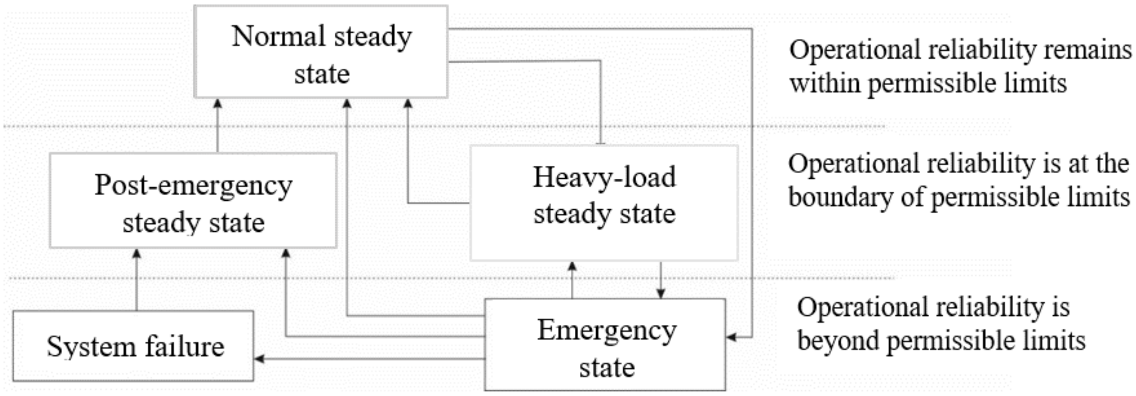

Fig. 1. Classification of the EPS steady states.

return to near-normal operating parameters. However, they may experience overloads. This state is similar to a heavy-load steady state regarding its operational parameters.

To prevent the EPS from transition to a less reliable state, a timely analysis of potential transitions is required through an operational reliability assessment. The EPS reliability assessment necessitates analyzing the largest possible number of potential operating states within the minimal time. Reliability indices [3, 4] are derived from the analysis to develop an appropriate set of control actions that boost operational reliability and preclude transitions to a degraded state.

The primary time-intensive procedure in the overall reliability assessment is the analysis of the EPS states. Reducing the time while maintaining the required accuracy enables an analysis of a larger number of potential scenarios, thereby enhancing the overall assessment fidelity. This reduction is achieved by employing steady-state sensitivity matrices [5, 6]. In this paper, sensitivity matrices are derived for the EPS mathematical model expressed in the vector power balance form [7–9]. This model is represented by a non-linear function. To ensure correct approximation of its underlying dependencies, it is necessary to assess the applicability of higher-order sensitivity matrices.

II. First-Order Steady-State Differential Models (Sensitivity Matrices) of Electric Power System

The mathematical model in power balance form, used to calculate the EPS steady states and to derive the sensitivity matrices (differential models), is as follows:

$$\bar{S} = \bar{\tilde{U}} Y_N U , \qquad (1)$$

where $\bar{S}$ is the $n \times 1$ complex conjugate vector of power at the EPS buses (n is the number of the EPS buses); $\bar{\tilde{U}}$ is the $n \times n$ diagonal complex conjugate matrix containing the elements of the vector $\bar{U}$ ($n \times 1$ vector of complex conjugate bus voltages) on its diagonal; Y_N is the $n \times n$ bus admittance matrix [7, 9]; U is the $n \times 1$ vector of complex values of voltages at EPS buses.

One of the most critical tasks in post-emergency state analysis for an operational reliability assessment is to establish a relationship between the changes in EPS bus voltages $dU_{\bar{S}}$ and power fluctuations $d\bar{S}$ (generator outages or load changes). Differentiating (1), while considering several specific features of the process, results in the following relationship:

$$dU_{\bar{S}} = (\bar{\tilde{U}}_0 Y_N)^{-1} d\bar{S} , \qquad (2)$$

where $\frac{dU_{\bar{S}}}{d\bar{S}} = (\bar{\tilde{U}}_0 Y_N)^{-1}$ is the $(n \times n)$ first-order sensitivity matrix, representing the dependence of voltage changes on power fluctuations at the EPS buses; $\bar{\tilde{U}}_0$ is the $n \times n$ diagonal complex conjugate matrix containing the elements of the vector $\bar{U}$, calculated at the base state, on its diagonal.

The sought voltage vector is determined as follows:

$$U \approx U_0 + dU_{\bar{S}} . \qquad (3)$$

Expression (3) represents the application of the Euler method for solving the ordinary differential equations (2). The sensitivity matrix $\frac{dU_{\bar{S}}}{d\bar{S}} = (\bar{\tilde{U}}_0 Y_N)^{-1}$ is formulated as a Cauchy problem (an initial value problem [10]). In the Euler method, the input perturbation acts as parameter increment. However, to improve the accuracy of the sought

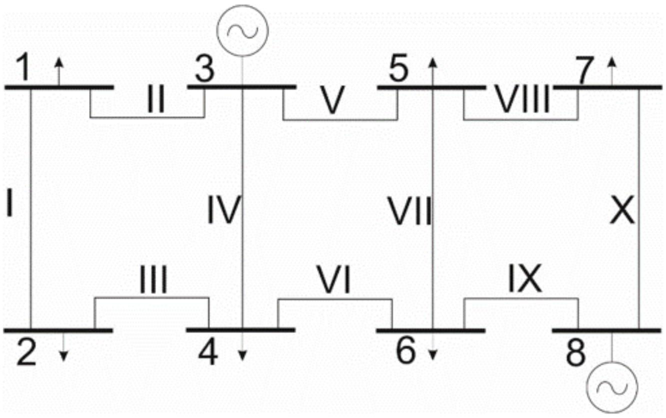

Fig. 2. Test EPS.

values, a multi-step approximation can be applied by dividing the perturbation into smaller increments. The recurrence formula for the case under consideration is as follows:

$$U^{n+1} \approx U^n + (\bar{\tilde{U}}_0 Y_N)^{-1} d\bar{S}^n , \qquad (4)$$

where U^{n+1} is the $(n \times 1)$ vector of output parameter values at step $n+1$ (the sought vector); U^n is the $n+1$ vector of output parameter values at step n (the known vector); $d\bar{S}^n$ is the step size.

While a multi-step approach enhances accuracy, it increases the time for additional calculations. Nevertheless, these calculations remain simple as they do not require derivative evaluation at each step.

III. Second-Order Steady-State Differential Models (Sensitivity Matrices) of Electric Power System

To derive the second-order differentials $d^2U_{\bar{S}}$, which represents the voltage sensitivity to bus power balance variations, equation (2) is differentiated. Furthermore, the fixed complex conjugate voltages are assumed as variables again, and the differential of the matrix function is obtained as follows [11]:

$$\begin{aligned} d^2U_{\bar{S}} &= -(\bar{\tilde{U}}Y_N)^{-1} d\bar{\tilde{U}}_{\bar{S}} Y_N (\bar{\tilde{U}}Y_N)^{-1} d\bar{S} = \\ &= -(\bar{\tilde{U}}Y_N)^{-1} (\bar{\tilde{U}}Y_N)^{-1} d\tilde{S}\bar{Y}_N (\bar{\tilde{U}}Y_N)^{-1} d\bar{S} = \\ &= -(\bar{\tilde{U}}Y_N)^{-2} d\tilde{S}\bar{Y}_N (\bar{\tilde{U}}Y_N)^{-1} d\bar{S}. \end{aligned} \qquad (5)$$

The second-order sensitivity matrix, representing the EPS bus voltage changes relative to power variations, cannot be derived from (5) because the function $U(S)$ is implicit.

In the case under consideration, the voltage variations, given events occurring in the EPS, are calculated as follows:

$$U \approx U_0 + dU_{\bar{S}} + 0.5 d^2 U_{\bar{S}} .$$

IV. Experimental Studies of First- and Second-Order EPS Steady-State Differential Models in Power Balance Form

The results of applying the first- and second-order differential models are demonstrated using a test case. The

Table 1. Input Data on Buses for the Base Steady-State Calculation of an Eight-Bus Electric Power System

Bus number	Bus type	*P*, MW	*Q*, MVAr	*V*, kV	δ, deg	U_a, kV	U_r, kV
1	*P*, *Q*	-50*	-20	–	–	–	–
2	*P*, *Q*	-10	-5	–	–	–	–
3	*P*, *V*	3×50 = 150	–	220	–	220	–
4	*P*, *Q*	-30	-15	–	–	–	–
5	*P*, *Q*	-80	-40	–	–	–	–
6	*P*, *Q*	-100	-25	–	–	–	–
7	*P*, *Q*	-60	-30	–	–	–	–
8	Slack *V*, δ	–	–	220	0	220	0

* "–" stands for power consumption.

TABLE 2. Input Data on Branches for the Base Steady-State Calculation of an Eight-Bus Electric Power System

№	Interconnected buses	Impedance, Ohm	Admittance, S
1	1–2	10 + *j*31	0.00942 – *j*0.02922
2	1–3	28 + *j*117	0.00193 – *j*0.00808
3	2–4	24 + *j*104	0.00211 – *j*0.00912
4	3–4	30 + *j*93	0.00314 – *j*0.00973
5	3–5	10 + *j*35	0.00755 – *j*0.02642
6	4–6	25 + *j*107	0.00207 – *j*0.00886
7	5–6	7 + *j*22	0.01313 – *j*0.04127
8	5–7	7 + *j*23	0.01211 – *j*0.03979
9	6–8	7 + *j*25	0.01039 – *j*0.03709
10	7–8	10 + *j*30	0.01 – *j*.03

TABLE 3. Results of the Base Steady-State Calculation (Calculated Operating Parameters are Shown in Bold)

Bus number	Bus type	*P*, MW	*Q*, MVAr	*V*, kV	δ, deg	U_a, kV	U_r, kV
1	*P*, *Q*	-50	-20	**201.68**	**-6.46**	**200.4**	**-22.7**
2	*P*, *Q*	-10	-5	**202.14**	**-6.22**	**200.96**	**-21.9**
3	*P*, *V*	3×50 = 150	**82.14**	220	**-0.48**	**219.99**	**-1.83**
4	*P*, *Q*	-30	-15	**207.29**	**-4.15**	**206.75**	**-14.99**
5	*P*, *Q*	-80	-40	**211.17**	**-2.9**	**210.9**	**-10.68**
6	*P*, *Q*	-100	-25	**211.72**	**-3**	**211.43**	**-11.08**
7	*P*, *Q*	-60	-30	**211.81**	**-2.43**	**211.62**	**-8.98**
8	Slack *V*, δ	**187.75**	**80.19**	220	0	220	0

test EPS is illustrated in Fig. 2. The system consists of eight buses and ten transmission lines connecting them.

Table 1 presents the initial data for buses to calculate the base steady state of the EPS. The input data on EPS branches for the calculation of the base steady state are provided in Table 2.

In these experimental studies, the shunt admittances of the transmission lines are neglected, without loss of generality.

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Further calculations were performed using reduced matrices and vectors, obtained by partitioning the slack bus. Based on the obtained voltages at the solution point at the EPS buses and the reactive power at the slack bus, the sensitivity matrix $\frac{dU_S}{d\bar{S}}$ is determined as follows:

$$\begin{bmatrix} 0.2 + j0.53 & 0.17 + j0.46 & 0.06 + j0.2 & 0.09 + j0.25 & 0.03 + j0.1 & 0.02 + j0.07 & 0.02 + j0.06 \\ 0.17 + j0.46 & 0.2 + j0.53 & 0.06 + j0.19 & 0.09 + j0.26 & 0.03 + j0.1 & 0.03 + j0.07 & 0.02 + j0.54 \\ 0.09 + j0.21 & 0.08 + j0.2 & 0.07 + j0.2 & 0.06 + j0.17 & 0.04 + j0.1 & 0.02 + j0.07 & 0.02 + j0.06 \\ 0.1 + j0.25 & 0.1 + j0.27 & 0.05 + j0.16 & 0.12 + j0.33 & 0.03 + j0.09 & 0.03 + j0.07 & 0.02 + j0.5 \\ 0.04 + j0.1 & 0.04 + j0.1 & 0.03 + j0.1 & 0.03 + j0.09 & 0.04 + j0.11 & 0.02 + j0.06 & 0.02 + j0.06 \\ 0.03 + j0.07 & 0.03 + j0.07 & 0.02 + j0.07 & 0.03 + j0.08 & 0.02 + j0.06 & 0.03 + j0.09 & 0.01 + j0.04 \\ 0.02 + j0.06 & 0.02 + j0.06 & 0.02 + j0.06 & 0.01 + j0.05 & 0.02 + j0.06 & 0.01 + j0.04 & 0.04 + j0.09 \end{bmatrix}$$

Generator outages occur frequently in the power systems and serve as one of the primary emergency events in reliability assessment. An outage of the 50 MW generator at bus 3 is simulated to test the model. The post-emergency state parameters, calculated using the ANARES software, are presented in Table 4 (U_{aA}, U_{rA}). The Table also presents the operating parameters (U_{aS}, U_{rS}) calculated for this scenario using the sensitivity matrix $\frac{dU_S}{d\bar{S}}$.

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TABLE 4. Post-Emergency Steady-State Parameters for a 50 MW Generator Outage at Bus 3, Calculated Using the ANARES Software and Sensitivity Matrix

Bus number	U_{aA}, kV	U_{rA}, kV	U_{aS}, kV	U_{rS}, kV	$\lvert U_{aA}-U_{aS}\rvert$, kV	$\lvert U_{rA}-U_{rS}\rvert$, kV	% U_a	% U_r
1	199.05	-33.03	197.48	-32.64	1.57	0.39	0.78	1.18
2	199.72	-31.83	198.27	-31.44	1.52	0.39	0.76	1.22
3	219.5	-14.83	216.59	-13.24	2.91	1.59	1.32	10.72
4	205.99	-23.8	204.41	-23.25	1.58	0.55	0.76	2.31
5	210.38	-16.3	209.31	-15.79	1.07	0.51	0.51	3.12
6	211.04	-14.6	210.44	-14.34	0.6	0.26	0.28	1.78
7	211.24	-12.13	210.68	-11.87	0.56	0.26	0.26	2.14
8	220	0	220	0	0	0	0	0

TABLE 5. Second-Order Differentials of Active and Reactive Voltage Components at EPS Buses for a 50 MW Generator Outage at Bus 3

Bus number	$d^2U_{a\bar{S}}$, kV	$d^2U_{r\bar{S}}$, kV	$U_{aA}^{(2)}$, kV	$U_{rA}^{(2)}$, kV
1	0.483	0.285	197.89	-32.83
2	0.465	0.272	198.58	-31.63
3	0.553	0.331	220	-13.47
4	0.402	0.232	204.75	-23.4
5	0.247	0.152	209.51	-15.89
6	0.158	0.095	210.58	-14.41
7	0.139	0.088	210.79	-11.93
8	0	0	220	0

The Table also presents the operating parameters (U_{aS}, U_{rS}) calculated for this scenario using the sensitivity matrix $\frac{dU_S}{d\bar{S}}$.

To compare the bus voltages obtained through direct software calculation (U_{aA}, U_{rA}) with those estimated using only the first-order differential (U_{aS}, U_{rS}), and both the first- and second-order differentials ($U_{aS}^{(2)}$, $U_{rS}^{(2)}$), the Euclidean norms of the resulting vectors were calculated. The norm of the voltage vector calculated using the ANARES software is 554.33. For the vectors incorporating the first-order differential and the combined first- and second-order differentials, the norm values stand at 550.62 and 551.11, respectively. Consequently, the incorporation of the second-order differentials significantly improves the accuracy of the EPS bus voltage estimations.

V. CONCLUSION

Operational reliability assessment of electrical power systems is essential for mitigating the risk of system failures. It requires analyzing a vast number of states involving potential equipment failures and other stochastic events within minimal time. The proposed method evaluates post-emergency steady states using sensitivity matrices (differential models). Specifically, this study explores the performance of the first- and second-order sensitivity matrices derived from the power balance equations. Findings from the test case demonstrate that the second-order sensitivity matrices significantly enhance computational accuracy.

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