On the Problem of Optimal Control of the Motion of Two-Link Planar Manipulator with Nonseparated Multipoint Intermediate Conditions

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***Abstract* — The study focuses on the problem of optimal control of the motion of a two-link planar manipulator on a fixed base with given initial and final conditions, nonseparated conditions for the values of the phase vector at intermediate times, and with a quality criterion given over the entire time interval. It is assumed that absolutely rigid links of the manipulator are interconnected by an ideal cylindrical hinge, and the similar hinge is used to attach the first link to the base. The optimal rules of changing the control moments are constructed, which allow the manipulator to move from a given initial state to a final one, satisfying nonseparated multipoint intermediate conditions. An application of the proposed approach is exemplified by constructed control functions and the corresponding motion with given nonseparated conditions for the values of the phase vector coordinates at some two intermediate times.**

***Index Terms*: two-link manipulator, optimal control, nonseparated multipoint conditions, phase constraints.**

I. Introduction

Problems of control and optimal control of dynamical systems with given constraints on the values of the coordinates of the phase vector at intermediate times arise in a number of problems important for applications. Similar problems, in particular, are encountered in the case of control and optimal control of manipulation robots, aircraft, technological processes, energy-saving control of thermal devices, and others [1–3]. Such a wide demand requires the development and design of modern (highly efficient) optimal control methods, which easily implement the control of the manipulator, leading to the desired movement. When studying the movements of manipulators and designing control systems, a mechanical model of a manipulator is usually used in the form of a system of absolutely rigid bodies (rods), which are connected with each other in series using ideal hinges [7–13]. Some important applied problems involve solving the problems of control and optimal control of the movement of manipulators as dynamic systems with nonseparated multipoint intermediate conditions. A characteristic feature of these problems is, along with the classical boundary (initial and final) conditions, the presence of nonseparated (nonlocal) conditions at several intermediate points of the considered interval. The study of these problems is of great importance for both theory and applications. Some issues of control and optimal control of linear dynamical systems with nonseparated multipoint intermediate conditions are examined, in particular, in [1–6].

This paper considers the problem of optimal control of the motion of a two-link planar manipulator on a fixed base with given initial and final conditions, nonseparated conditions for the values of the phase vector at intermediate times, and with a quality criterion given over the entire time interval. Based on the mathematical model of a two-link planar manipulator in the form of Lagrange equations of the second kind [14], in which the main moments are controls, we have constructed explicit forms of the optimal control action and the corresponding motion using the method of moment problems [15].

II. Mathematical Model of the Manipulator and Problem Statement

We consider a two-link manipulator (see Fig. 1) consisting of two absolutely rigid bodies (links) G_1, G_2 connected by hinge O_2. Body G_1 is connected with a fixed base by means of hinge O_1. The hinges are ideal, cylindrical, and their axes are parallel to each other. The

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http://dx.doi.org/10.38028/esr.2022.04.0005
Received October 26, 2022. Revised November 11, 2022.
Accepted November 03, 2022. Available online January 30, 2023.

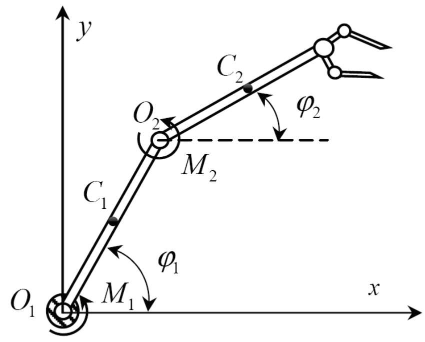

Fig. 1. Two-link manipulator.

system moves in a horizontal plane perpendicular to the hinge axes O_1, O_2.

Each link of the manipulator is an absolutely rigid homogeneous rod of length L. It is assumed that link G_2 includes the executive body (grip), i.e., the mass of the gripper is neglected and the dynamic characteristics are not considered separately. Manipulator is controlled by two independent drives D_1, D_2. Drive D_1 carries out the interaction of body G_1 with the base, and drive D_2 carries out the interaction between link G_1 and link G_2 of the manipulator. The main force vectors generated by drives D_1, D_2 are equal to zero, and the main moments relative to the hinge axes O_1, O_2 are equal to M_1, M_2, respectively. Values M_1, M_2 are taken as control functions in the considered model of the manipulator. It is also assumed that control functions belong to the class of piecewise continuous functions. We do not take into account the action of other forces.

Let us introduce a fixed Cartesian coordinate system O_1XY with the origin on the hinge axis O_1 in the considered plane. Let us denote by φ_1, φ_2 the angles between the horizontal axis and the first and second links, respectively; I_1, I_2 are the moments of inertia of bodies G_1, G_2 relative to the corresponding axes; $L_1 = |O_1O_2|$ is the distance between hinge axes, $L_2 = |O_2C_2|$ is the distance from axis O_2 to the center of gravity C_2 for link G_2.

The kinetic energy of the two links is equal to

$$K = \frac{1}{2}\left(I_1 + m_2L_1^2\right)\dot{\varphi}_1^2 + \frac{1}{2}\left(I_2 + m_2L_2^2\right)\dot{\varphi}_2^2 + \\ + m_2L_1L_2\cos\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_1\dot{\varphi}_2.$$

The equations of motion of the considered manipulator in the form of Lagrange differential equations of the second kind have the form:

$$\begin{gathered}\left(I_1 + m_2L_1^2\right)\ddot{\varphi}_1 + m_2L_1L_2\cos\left(\varphi_1 - \varphi_2\right)\ddot{\varphi}_2 + \\ + m_2L_1L_2\sin\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_2^2 = M_1 - M_2, \\ (I_2 + m_2L_2^2)\ddot{\varphi}_2 + m_2L_1L_2\cos\left(\varphi_1 - \varphi_2\right)\ddot{\varphi}_1 - \\ - m_2L_1L_2\sin\left(\varphi_1 - \varphi_2\right)\dot{\varphi}_1^2 = M_2.\end{gathered} \tag{1.1}$$

It is assumed that the center of mass of the second link is located on the axis of hinge O_2, connecting with the first link, which corresponds to the static balance of the second link of the manipulator. In this case, assuming that $|O_2C_2| = L_2 = 0$, equation (1.1) has the form

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u_1,\ \dot{x}_3 = x_4,\ \dot{x}_4 = u_2, \tag{1.2}$$

where

$$x_1 = \left(I_1 + m_2L_1^2\right)\varphi_1,\ x_2 = \left(I_1 + m_2L_1^2\right)\dot{\varphi}_1,$$

$$x_3 = \left(I_2 + m_2L_2^2\right)\varphi_2,\ x_4 = \left(I_2 + m_2L_2^2\right)\dot{\varphi}_2.$$

Control functions u_1 and u_2 have the form

$$u_1 = M_1 - M_2,\ u_2 = M_2,$$

where M_1, M_2 are the main moments relative to the hinge axes.

Let the initial and final states of system (1.2) be given

$$\begin{gathered} x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0), x_4(t_0))^T, \\ x(T) = (x_1(T), x_2(T), x_3(T), x_4(T))^T, \end{gathered} \tag{1.3}$$

and, at some fixed intermediate time instants

$$0 \le t_0 < t_1 < t_2 < t_3 = T,$$

nonseparated (nonlocal) multipoint intermediate conditions

$$\sum_{k=1}^{2} F_k x(t_k) = \alpha \tag{1.4}$$

be given, where α is a two-dimensional column vector, F_k are (2×4)-dimensional matrices ($k = 1, 2$), whose elements are real numbers [4].

In general, for some cases, it can be assumed that at intermediate times t_k ($k = 1, 2$) not all values of the coordinates of the phase vector $x(t_k)$ are present in (1.4), but only some of them. In such cases, we will assume the corresponding elements of the matrix F_k to equal zero.

System (1.2) with multipoint intermediate condition (1.4) on the time interval $[t_0, T]$ is completely controllable [2, 14].

The optimal control problem for system (1.2) with nonseparated multipoint intermediate conditions (1.4) can be formulated as follows.

Find the optimal control actions $u_1^0(t)$ ***and*** $u_2^0(t)$***,*** $t \in [t_0, T]$***, which transfer the solution to system (1.2) from the initial state*** $x(t_0)$ ***to the final state*** $x(T)$***, thereby ensuring satisfaction of the nonseparated multipoint intermediate condition (1.4) and having the smallest possible value of the quality criterion*** æ$[u^0]$**:**

$$æ[u] = \left(\int_{t_0}^{T} (u_1^2 + u_2^2)dt\right)^{\frac{1}{2}}. \tag{1.5}$$

III. Solution to the Problem

To solve the problem, we write the solution to Eq. (1.2) following from the initial state $x(t_0)$, and by substituting the values $x(t_k)$ into (1.4) for the time instants $t = t_k$ ($k = 1, 2$), obtain the following relations:

$$\sum_{k=1}^{2} F_k X[t_k, t_0]x(t_0) + \sum_{k=1}^{2}\int_{t_0}^{t_k} F_k X[t_k, \tau]Bu(\tau)d\tau = \alpha. \tag{2.1}$$

For a finite time $t = T$, we have

$$x(T) = X[T, t_0]x(t_0) + \int_{t_0}^{T} X[T, \tau]Bu(\tau)d\tau, \tag{2.2}$$

where $X[t, \tau]$ denotes the normalized fundamental matrix of the solution to the homogeneous part of equation (1.2). The matrices B and $X[t, \tau]$ have the following form:

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X[t,\tau] = \begin{pmatrix} x_{11}(t,\tau) & x_{12}(t,\tau) & 0 & 0 \\ 0 & x_{22}(t,\tau) & 0 & 0 \\ 0 & 0 & x_{33}(t,\tau) & x_{34}(t,\tau) \\ 0 & 0 & 0 & x_{44}(t,\tau) \end{pmatrix},$$

where

$$x_{11}(t, \tau) = x_{22}(t, \tau) = x_{33}(t, \tau) = x_{44}(t, \tau) = 1;$$
$$x_{12}(t, \tau) = x_{34}(t, \tau) = t - \tau. \tag{2.3}$$

Using the approaches given in [2, 4], from (2.1) and (2.2) we obtain the following integral relation

$$\int_{t_0}^{T} H[t]u(t)dt = \eta(t_0,....,T), \tag{2.4}$$

where the following notation

$$H[t] = \begin{pmatrix} F(t)B \\ X[T,t]B \end{pmatrix},$$

$$\eta(t_0,....,T) = \begin{pmatrix} \alpha - Fx(t_0) \\ x(T) - X[T,t_0]x(t_0) \end{pmatrix},$$

$$F(t) = \sum_{k=1}^{2} F_k[t] = \sum_{k=1}^{2} F_k X[t_k,t],$$

$$F = \sum_{k=1}^{2} F_k X[t_k,t_0] = F(t_0),$$

$$F_k[t] = \begin{cases} F_k X[t_k,t], \text{ for } t_0 \le t \le t_k, \\ 0, \quad \text{for } t_k < t \le t_{m+1} = T, \end{cases} \quad k = 1,2, \tag{2.5}$$

is accepted. Here $H[t]$ is a (6×2) block matrix, the known matrices $F(t)$ and F have dimension (2×4), and η is a (6×1)-dimensional known column vector.

For system (1.2) with nonseparated multipoint intermediate condition (1.4) to be completely controllable on the interval $[t_0, T]$, it is necessary and sufficient that the column vectors of the matrix $H[t]$ be linearly independent on this interval.

Let nonseparated intermediate values (1.4) have the form:

$$x_1(t_1) + x_3(t_1) + x_1(t_2) + x_3(t_2) = \alpha_1,$$
$$x_2(t_1) + x_4(t_1) + x_2(t_2) + x_4(t_2) = \alpha_2, \tag{2.6}$$

i.e., $\alpha = (\alpha_1, \alpha_2)^T$, $F_1 = F_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Substituting the expressions for matrices F_1, F_2 and the fundamental matrix of solution $X[t, \tau]$ into formula (2.5), we have

$$F(\tau) = F_1 X[t_1,\tau] + F_2 X[t_2,\tau] = \begin{pmatrix} f_{11}(\tau) & f_{12}(\tau) & f_{13}(\tau) & f_{14}(\tau) \\ f_{21}(\tau) & f_{22}(\tau) & f_{23}(\tau) & f_{24}(\tau) \end{pmatrix}, \tag{2.7}$$

where

$$f_{11}(\tau) = x_{11}(t_1,\tau) + x_{11}(t_2,\tau);$$
$$f_{12}(\tau) = x_{12}(t_1,\tau) + x_{12}(t_2,\tau);$$
$$f_{13}(\tau) = x_{33}(t_1,\tau) + x_{33}(t_2,\tau);$$
$$f_{14}(\tau) = x_{34}(t_1,\tau) + x_{34}(t_2,\tau);$$
$$f_{22}(\tau) = x_{22}(t_1,\tau) + x_{22}(t_2,\tau);$$
$$f_{24}(\tau) = x_{44}(t_1,\tau) + x_{44}(t_2,\tau);$$
$$f_{21}(\tau) = f_{23}(\tau) = 0.$$

Therefore, matrix $H[t]$ will be presented in the form:

$$H[\tau] = \begin{pmatrix} x_{12}(t_1,\tau) + x_{12}(t_2,\tau) & x_{34}(t_1,\tau) + x_{34}(t_2,\tau) \\ x_{22}(t_1,\tau) + x_{22}(t_2,\tau) & x_{44}(t_1,\tau) + x_{44}(t_2,\tau) \\ x_{12}(T,\tau) & 0 \\ x_{22}(T,\tau) & 0 \\ 0 & x_{34}(T,\tau) \\ 0 & x_{44}T(t_2,\tau) \end{pmatrix}.$$

According to (2.4)–(2.6), we will have the following integral relations:

$$\int_{t_0}^{T} \left[h_{11}(\tau)u_1 + h_{12}(\tau)u_2 \right] d\tau = \eta_1,$$

$$\int_{t_0}^{T} \left[h_{21}(\tau)u_1 + h_{22}(\tau)u_2 \right] d\tau = \eta_2,$$

$$\int_{t_0}^{T} h_{31}(\tau)u_1 d\tau = \eta_3, \quad \int_{t_0}^{T} h_{41}(\tau)h_1(\tau)\, d\tau = \eta_4,$$

$$\int_{t_0}^{T} h_{52}(\tau)u_2 d\tau = \eta_5, \quad \int_{t_0}^{T} h_{62}(\tau)u_2 d\tau = \eta_6, \tag{2.8}$$

where the following notation

$$h_{11}(\tau) = x_{12}(t_1,\tau) + x_{12}(t_2,\tau);$$
$$h_{12}(\tau) = x_{34}(t_1,\tau) + x_{34}(t_2,\tau);$$
$$h_{21}(\tau) = x_{22}(t_1,\tau) + x_{22}(t_2,\tau);$$
$$h_{22}(\tau) = x_{44}(t_1,\tau) + x_{44}(t_2,\tau);$$
$$h_{31}(\tau) = x_{12}(T,\tau);\ h_{41}(\tau) = x_{22}(T,\tau);$$
$$h_{52}(\tau) = x_{34}(T,\tau);\ h_{62}(\tau) = x_{44}(T,\tau);$$
$$\eta_1 = \alpha_1 - 2[x_1(t_0) + x_3(t_0)] -$$
$$- (t_1 + t_2 - 2t_0)[x_2(t_0) + x_4(t_0)]; \tag{2.9}$$
$$\eta_2 = \alpha_2 - 2[x_2(t_0) + x_4(t_0)];$$
$$\eta_3 = x_1(T) - x_1(t_0) - (T - t_0)\, x_2(t_0);$$
$$\eta_4 = x_2(T) - x_2(t_0);$$
$$\eta_5 = x_3(T) - x_3(t_0) - (T - t_0)\, x_4(t_0);$$
$$\eta_6 = x_4(T) - x_4(t_0)$$

is accepted.

For a given performance criterion æ$[u]$, the optimal control problem with integral condition (2.4) is a conditional extremum problem, where the minimum of the functional æ$[u]$ must be determined under conditions (2.4).

The left-hand side of condition (2.4) is a linear operation generated by function $u(t)$ on the time interval $[t_0, T]$, and the functional is the norm of a normed linear space. Then the optimal control action $u^0(t)$, $[t_0, T]$, minimizing the functional æ$[u]$ and satisfying condition (2.4) must be constructed according to the algorithm for solving optimal control problems using the moment problem method [15]. To solve the problem of moments (1.5) and ((2.8)), following [15], we need to find the quantities l_i, $i = 1,\ldots,6$, related by condition

$$\sum_{i=1}^{6} l_i \eta_i = 1, \tag{2.10}$$

for which

$$(\rho_0)^2 = \min_{(2.9)} \int_0^T \left[h_1^2(\tau) + h_2^2(\tau) \right] d\tau, \tag{2.11}$$

where

$$h_1(\tau) = l_1 h_{11}(\tau) + l_2 h_{21}(\tau) + l_3 h_{31}(\tau) + l_4 h_{41}(\tau),$$
$$h_2(\tau) = l_1 h_{12}(\tau) + l_2 h_{22}(\tau) + l_5 h_{52}(\tau) + l_6 h_{62}(\tau), \tag{2.12}$$

To determine the quantities l_i^0, $i = 1,\ldots,6$, minimizing (2.11), we apply the method of indefinite Lagrange multipliers. Let us introduce the function

$$f = \int_{t_0}^{T} \left[\left(h_1(\tau)\right)^2 + \left(h_2(\tau)\right)^2 \right] d\tau + \lambda \left[\sum_{i=1}^{6} l_i \eta_i - 1 \right],$$

where λ is the indefinite Lagrange multiplier. Based on this method, calculating the derivatives of function f with respect to l_i, $i = 1,\ldots,6$ and equating them to zero, we obtain the following system of integral relations:

$$\int_{t_0}^{T} \left[h_{11}(\tau) h_1(\tau) + h_{12}(\tau) h_2(\tau) \right] d\tau = -\frac{\lambda}{2} \eta_1,$$

$$\int_{t_0}^{T} \left[h_{21}(\tau) h_1(\tau) + h_{22}(\tau) h_2(\tau) \right] d\tau = -\frac{\lambda}{2} \eta_2,$$

$$\int_{t_0}^{T} h_{31}(\tau) h_1(\tau) d\tau = -\frac{\lambda}{2} \eta_3, \quad \int_{t_0}^{T} h_{41}(\tau) h_1(\tau) d\tau = -\frac{\lambda}{2} \eta_4, \tag{2.13}$$

$$\int_{t_0}^{T} h_{52}(\tau) h_2(\tau) d\tau = -\frac{\lambda}{2} \eta_5, \quad \int_{t_0}^{T} h_{62}(\tau) h_2(\tau) d\tau = -\frac{\lambda}{2} \eta_6.$$

Given the notation (2.12), equations (2.13) can be written in the form of the following algebraic equations:

$$a_{11} l_1 + a_{12} l_2 + a_{13} l_3 + a_{14} l_4 + a_{15} l_5 + a_{16} l_6 = -(\lambda/2)\eta_1,$$
$$a_{21} l_1 + a_{22} l_2 + a_{23} l_3 + a_{24} l_4 + a_{25} l_5 + a_{26} l_6 = -(\lambda/2)\eta_2,$$
$$a_{31} l_1 + a_{32} l_2 + a_{33} l_3 + a_{34} l_4 = -(\lambda/2)\eta_3,$$
$$a_{41} l_1 + a_{42} l_2 + a_{43} l_3 + a_{44} l_4 = -(\lambda/2)\eta_4, \tag{2.14}$$
$$a_{51} l_1 + a_{52} l_2 + a_{55} l_5 + a_{56} l_6 = -(\lambda/2)\eta_5,$$
$$a_{61} l_1 + a_{62} l_2 + a_{65} l_5 + a_{66} l_6 = -(\lambda/2)\eta_6.$$

The following notation is used here:

$$a_{11} = \int_{t_0}^{T} \left[\left(h_{11}(\tau)\right)^2 + \left(h_{12}(\tau)\right)^2 \right] d\tau =$$

$$= \int_{t_0}^{t_1} \Big[\left(x_{12}(t_1,\tau)\right)^2 + 2x_{12}(t_1,\tau) x_{12}(t_2,\tau) +$$
$$+ \left(x_{34}(t_1,\tau)\right)^2 + 2x_{34}(t_1,\tau) x_{34}(t_2,\text{t}) \Big] d\tau +$$
$$+ \int_{t_0}^{t_2} \left[\left(x_{12}(t_2,\tau)\right)^2 + \left(x_{34}(t_2,\tau)\right)^2 \right] d\tau,$$

$$a_{12} = a_{21} = \int_{t_0}^{T} \left[h_{11}(\tau) h_{21}(\tau) + h_{12}(\tau) h_{22}(\tau) \right] d\tau =$$

$$= \int_{t_0}^{t_1} \Big[x_{12}(t_1,\tau) x_{22}(t_1,\tau) + x_{12}(t_1,\tau) x_{22}(t_2,\tau) +$$
$$+ x_{12}(t_2,\tau) x_{22}(t_1,\tau) + x_{34}(t_1,\tau) x_{44}(t_1,\tau) +$$
$$+ x_{34}(t_1,\tau) x_{44}(t_2,\tau) + x_{34}(t_2,\tau) x_{44}(t_1,\tau) \Big] d\tau +$$
$$+ \int_{t_0}^{t_2} \left[x_{12}(t_2,\tau) x_{22}(t_2,\tau) + x_{34}(t_2,\tau) x_{44}(t_2,\tau) \right] d\tau,$$

$$a_{13} = a_{31} = \int_{t_0}^{T} h_{11}(\tau) h_{31}(\tau) d\tau =$$

$$= \int_{t_0}^{t_1} x_{12}(t_1,\tau) x_{12}(T,\tau) d\tau + \int_{t_0}^{t_2} x_{12}(t_2,\tau) x_{12}(T,\tau) d\tau,$$

$$a_{14} = a_{41} = \int_{t_0}^{T} h_{11}(\tau) h_{41}(\tau) d\tau =$$

$$= \int_{t_0}^{t_1} x_{12}(t_1,\tau) x_{22}(T,\tau) d\tau + \int_{t_0}^{t_2} x_{12}(t_2,\tau) x_{22}(T,\tau) d\tau,$$

$$a_{15} = a_{51} = \int_{t_0}^{T} h_{12}(\tau) h_{52}(\tau) d\tau =$$

$$= \int_{t_0}^{t_1} x_{34}(t_1,\tau) x_{34}(T,\tau) d\tau + \int_{t_0}^{t_2} x_{34}(t_2,\tau) x_{34}(T,\tau) d\tau,$$

$$a_{16} = a_{61} = \int_{t_0}^{T} h_{12}(\tau) h_{62}(\tau) d\tau =$$

$$= \int_{t_0}^{t_1} x_{34}(t_1,\tau) x_{44}(T,\tau) d\tau + \int_{t_0}^{t_2} x_{34}(t_2,\tau) x_{44}(T,\tau) d\tau,$$

$$a_{22} = \int_{t_0}^{T} \left[\left(h_{21}(\tau)\right)^2 + \left(h_{22}(\tau)\right)^2 \right] d\tau =$$

$$= \int_{t_0}^{t_1} \Big[\left(x_{22}(t_1,\tau)\right)^2 + 2x_{22}(t_1,\tau) x_{22}(t_2,\tau) +$$
$$+ \left(x_{44}(t_1,\tau)\right)^2 + 2x_{44}(t_1,\tau) x_{44}(t_2,\tau) \Big] d\tau$$
$$+ \int_{t_0}^{t_2} \left[\left(x_{22}(t_2,\tau)\right)^2 + \left(x_{44}(t_2,\tau)\right)^2 \right] d\tau,$$

$$a_{23} = a_{32} = \int_{t_0}^{T} h_{21}(\tau) h_{31}(\tau) d\tau =$$

$$=\int_{t_0}^{t_1} x_{22}(t_1,\tau)x_{12}(T,\tau)d\tau+\int_{t_0}^{t_2} x_{22}(t_2,\tau)x_{12}(T,\tau)d\tau,$$

$$a_{24}=a_{42}=\int_{t_0}^{T} h_{21}(\tau)h_{41}(\tau)d\tau=$$

$$=\int_{t_0}^{t_1} x_{22}(t_1,\tau)x_{22}(T,\tau)d\tau+\int_{t_0}^{t_2} x_{22}(t_2,\tau)x_{22}(T,\tau)d\tau,$$

$$a_{25}=a_{52}=\int_{t_0}^{T} h_{22}(\tau)h_{52}(\tau)d\tau=$$

$$=\int_{t_0}^{t_1} x_{44}(t_1,\tau)x_{34}(T,\tau)d\tau+\int_{t_0}^{t_2} x_{44}(t_2,\tau)x_{34}(T,\tau)d\tau,$$

$$a_{26}=a_{62}=\int_{t_0}^{T} h_{22}(\tau)h_{62}(\tau)d\tau=$$

$$=\int_{t_0}^{t_1} x_{44}(t_1,\tau)x_{44}(T,\tau)d\tau+\int_{t_0}^{t_2} x_{44}(t_2,\tau)x_{44}(T,\tau)d\tau,$$

$$a_{33}=\int_{t_0}^{T}\left(h_{31}(\tau)\right)^2 d\tau=\int_{t_0}^{T}\left(x_{12}(T,\tau)\right)^2 d\tau,$$

$$a_{34}=a_{43}=\int_{t_0}^{T} h_{31}(\tau)h_{41}(\tau)d\tau=\int_{t_0}^{T} x_{12}(T,\tau)x_{22}(T,\tau)d\tau,$$

$$a_{44}=\int_{t_0}^{T}\left(h_{41}(\tau)\right)^2 d\tau=\int_{t_0}^{T}\left(x_{22}(T,\tau)\right)^2 d\tau,$$

$$a_{55}=\int_{t_0}^{T}\left(h_{52}(\tau)\right)^2 d\tau=\int_{t_0}^{T}\left(x_{34}(T,\tau)\right)^2 d\tau,$$

$$a_{56}=a_{65}=\int_{t_0}^{T} h_{52}(\tau)h_{62}(\tau)d\tau=\int_{t_0}^{T} x_{34}(T,\tau)x_{44}(T,\tau)d\tau,$$

$$a_{66}=\int_{t_0}^{T}\left(h_{62}(\tau)\right)^2 d\tau=\int_{t_0}^{T}\left(x_{44}(T,\tau)\right)^2 d\tau.$$

Adding condition (2.10) to the obtained equations (2.14), we obtain a closed system of seven algebraic equations for the same number of unknown quantities l_i, $i = 1,\ldots,6$, and λ. Let the quantities l_i^0, $i = 1,\ldots,6$, and λ_0 be the solution to this closed system of algebraic equations. Then, according to (2.11), (2.12), we have

$$h_1^0(\tau)=l_1^0h_{11}(\tau)+l_2^0h_{21}(\tau)+l_3^0h_{31}(\tau)+l_4^0h_{41}(\tau),$$
$$h_2^0(\tau)=l_1^0h_{12}(\tau)+l_2^0h_{22}(\tau)+l_5^0h_{52}(\tau)+l_6^0h_{62}(\tau),\quad (2.15)$$
$$(\rho_0)^2=\int_{t_0}^{T}\left[\left(h_1^0(\tau)\right)^2+\left(h_2^0(\tau)\right)^2\right]d\tau.$$

Following [15], the optimal control actions can be represented as:

$$u_1^0(t)=\frac{1}{\rho_0^2}h_1^0(t)\,;\;u_2^0(t)=\frac{1}{\rho_0^2}h_2^0(t)\,.$$

Taking into account the notation (2.15), the optimal control actions are represented as follows:

$$u_1^0(t)=\begin{cases}\frac{1}{\rho_0^2}[l_1^0\left(x_{12}(t_1,\tau)+x_{12}(t_2,\tau)\right)+l_2^0\left(x_{22}(t_1,\tau)+x_{22}(t_2,\tau)\right)+\\ +l_3^0x_{12}(T,\tau)+l_4^0x_{22}(T,\tau)], & t\in[t_0,t_1),\\ \frac{1}{\rho_0^2}\left[l_1^0x_{12}(t_2,\tau)+l_2^0x_{22}(t_2,\tau)+l_3^0x_{12}(T,\tau)+l_4^0x_{22}(T,\tau)\right], & \\ & t\in[t_1,t_2),\\ \frac{1}{\rho_0^2}\left[l_3^0x_{12}(T,\tau)+l_4^0x_{22}(T,\tau)\right], & t\in[t_2,T],\end{cases}$$

$$u_2^0(t)=\begin{cases}\frac{1}{\rho_0^2}[l_1^0\left(x_{34}(t_1,\tau)+x_{34}(t_2,\tau)\right)+l_2^0\left(x_{44}(t_1,\tau)+x_{44}(t_2,\tau)\right)+\\ +l_5^0x_{34}(T,\tau)+l_6^0x_{44}(T,\tau), & t\in[t_0,t_1),\\ \frac{1}{\rho_0^2}\left[l_1^0x_{34}(t_2,\tau)+l_2^0x_{44}(t_2,\tau)+l_5^0x_{34}(T,\tau)+l_6^0x_{44}(T,\tau)\right], & \\ & t\in[t_1,t_2),\\ \frac{1}{\rho_0^2}\left[l_5^0x_{34}(T,\tau)+l_6^0x_{44}(T,\tau)\right], & t\in[t_2,T],\end{cases}$$

or, given (2.3), they can have the form:

$$u_1^0(t)=\begin{cases}\frac{1}{\rho_0^2}\left[l_1^0(t_1+t_2-2\tau)+2l_2^0+l_3^0(T-\tau)+l_4^0\right], & \\ & t\in[t_0,t_1),\\ \frac{1}{\rho_0^2}\left[l_1^0(t_2-\tau)+l_2^0+l_3^0(T-\tau)+l_4^0\right], & t\in[t_1,t_2),\\ \frac{1}{\rho_0^2}\left[l_3^0(T-\tau)+l_4^0\right], & t\in[t_2,T],\end{cases}$$

$$u_2^0(t)=\begin{cases}\frac{1}{\rho_0^2}\left[l_1^0(t_1+t_2-2\tau)+2l_2^0+l_5^0(T-\tau)+l_6^0\right], & \\ & t\in[t_0,t_1),\\ \frac{1}{\rho_0^2}\left[l_1^0(t_2-\tau)+l_2^0+l_5^0(T-\tau)+l_6^0\right], & t\in[t_1,t_2),\\ \frac{1}{\rho_0^2}\left[l_5^0(T-\tau)+l_6^0\right], & t\in[t_2,T].\end{cases}$$

Substituting the expression for the optimal control action into (1.2) and integrating these equations, we obtain the optimal motion on each time interval.

IV. Example

Let some fixed intermediate times $0 \le t_0 < t_1 < t_2 = T$, and $t_0 = 0$; $t_1 = 2$; $t_2 = 3$; $T = 4$ be given. The initial and final states for phase vector $x = (x_1, x_2, x_3, x_4)^T$ will be $x(0) = (0, 0, 0, 0)^T$, $x(4) = (5, 0, 4, 1)^T$.

According to formula (2.9), assuming that $\alpha_1 = 3$, $\alpha_2 = 2$, we obtain the following value for the constant vector η:

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)^T = (3, 2, 5, 0, 4, 1)^T.$$

Further, carrying out the corresponding calculations of the integrals for the coefficients of the system of equations (2.14), we obtain

$a_{11} = 42$, $a_{12} = a_{21} = 25$, $a_{13} = a_{31} = a_{15} = a_{51} = 121/6$,
$a_{14} = a_{41} = a_{16} = a_{61} = 13/2$, $a_{22} = 18$,
$a_{23} = a_{32} = a_{25} = a_{52} = 27/2$, $a_{24} = a_{42} = a_{26} = a_{62} = 5$,

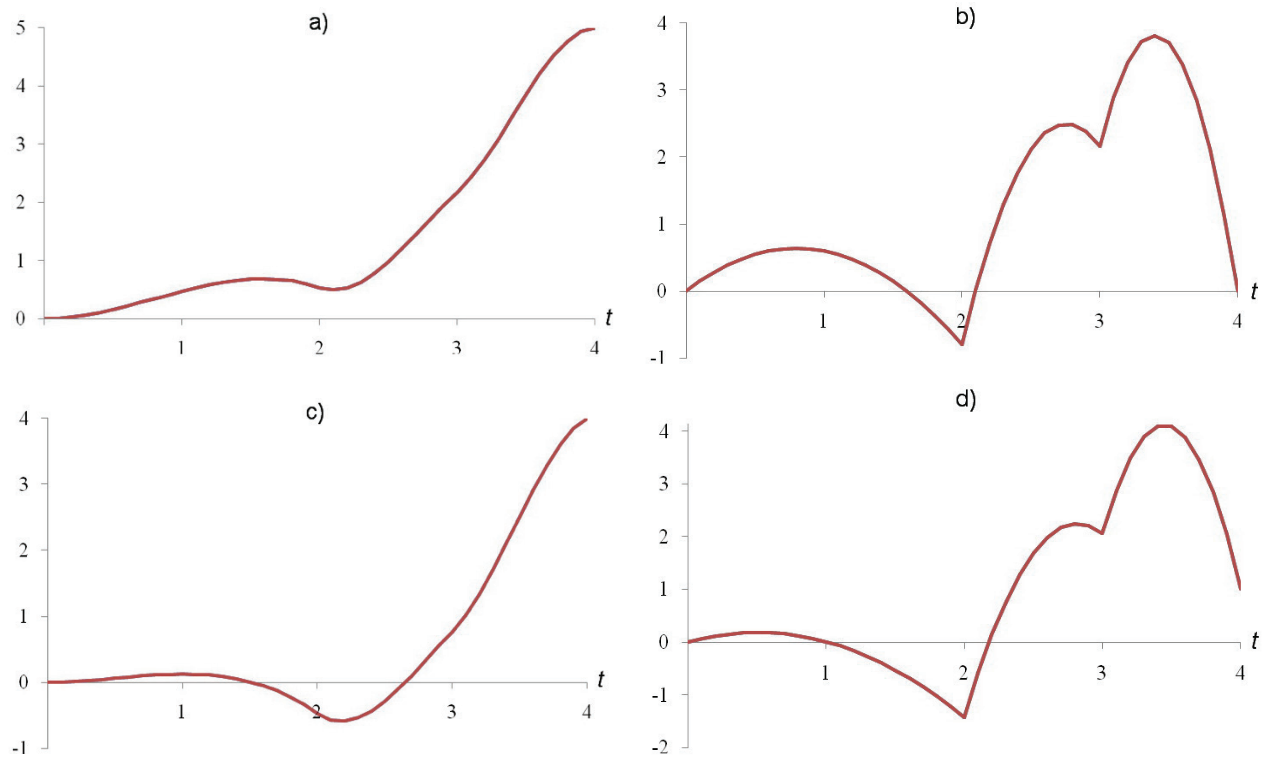

Fig. 2. Graphs of the vector function of the optimal movement** $x^0(t)$ **at** $t\in[0,4]$ **by coordinates: a)** $x_1^0(t)$ **; b)** $x_2^0(t)$ **; c)** $x_3^0(t)$ **; d)** $x_4^0(t)$ **.

$$a_{33} = 64/3,\ a_{44} = 4,\ a_{34} = a_{43} = a_{56} = a_{65} = 8,$$
$$a_{55} = 64/3,\ a_{66} = 4.$$

Solving the system of algebraic equations (2.14) with the obtained numerical values of the coefficients for l_i^0, $i = 1,\ldots,6$, and λ_0, we obtain the following values:

$$l_1=-\frac{18528}{243299},\ l_2=-\frac{21696}{243299},\ l_3=\frac{40965}{243299},$$
$$l_4=-\frac{24702}{243299},\ l_5=\frac{39867}{243299},$$
$$l_6=-\frac{22018}{243299},\ \lambda=-\frac{3904}{243299}.$$

Based on the notation (2.5), the optimal functions $h_1^0(\tau)$, $h_2^0(\tau)$, $\tau\in[t_0,T]$ are represented as follows:

$$h_1^0(\tau)=\begin{cases} l_1^0\left(t_1+t_2-2\tau\right)+2l_2^0+l_3^0\left(T-\tau\right)+l_4^0, \\ \qquad\qquad t\in\left[t_0,t_1\right), \\ l_1^0\left(t_2-\tau\right)+l_2^0+l_3^0\left(T-\tau\right)+l_4^0,\ t\in\left[t_1,t_2\right), \\ l_3^0\left(T-\tau\right)+l_4^0, \qquad t\in\left[t_2,T\right], \end{cases}$$

$$h_2^0(\tau)=\begin{cases} l_1^0\left(t_1+t_2-2\tau\right)+2l_2^0+l_5^0\left(T-\tau\right)+l_6^0, \\ \qquad\qquad t\in\left[t_0,t_1\right), \\ l_1^0\left(t_2-\tau\right)+l_2^0+l_5^0\left(T-\tau\right)+l_6^0,\ t\in\left[t_1,t_2\right), \\ l_5^0\left(T-\tau\right)+l_6^0, \qquad t\in\left[t_2,T\right]. \end{cases}$$

Now, calculating the value of $(\rho_0)^2$ according to formula (2.15), we obtain

$$(\rho_0)^2=\frac{1952}{243299}.$$

Further, for the components of the vector of optimal control action $u^0(t)$, $t\in[t_0,T]$, we will have explicit expressions in the following form:

$$u_1^0(t)=\begin{cases} 1.6014-2.0026t, & t\in[0,2], \\ 31.6998-11.4944t, & t\in(2,3], \\ 71.2899-20.9862t, & t\in(3,4], \end{cases}$$

$$u_2^0(t)=\begin{cases} 0.7264-1.4401t, & t\in[0,2], \\ 30.8248-10.9319t, & t\in(2,3], \\ 70.4149-20.4237t, & t\in(3,4]. \end{cases}$$

If we substitute the obtained expressions for the optimal control into (1.2) and integrate these equations, then we obtain the optimal motion on each time interval in the following form:

$$x_1^0(t)=\begin{cases} 0.8007t^2-0.3338t^3, \qquad t\in[0,2], \\ 15.8499t^2-1.9157t^3+34.8852-41.2131t, \\ \qquad\qquad t\in(2,3], \\ 35.6449t^2-3.4977t^3+127.6147-117.2705t, \\ \qquad\qquad t\in(3,4], \end{cases}$$

$$x_2^0(t)=\begin{cases} 1.6014t-1.0013t^2, & t\in[0,2], \\ 31.6998t-5.7472t^2-41.2131, & t\in(2,3], \\ 71.2899t-10.4930t^2-117.2705, & t\in(3,4], \end{cases}$$

$$x_3^0(t)=\begin{cases} 0.3632t^2-0.2400t^3, & t\in[0,2],\\ 15.4124t^2-1.8219t^3+34.8852-41.2131t, & \\ & t\in(2,3],\\ 35.2075t^2-3.4039t^3+127.6147- & \\ \quad -117.2705t, & t\in(3,4], \end{cases}$$

$$x_4^0(t)=\begin{cases} 0.7264t-0.7200t^2, & t\in[0,2],\\ 30.8248t-5.4659t^2-41.2131, & t\in(2,3],\\ 70.4149t-10.2118t^2-117.2705, & t\in(3,4]. \end{cases}$$

Figure 2 shows a graphical view of the vector-function of the optimal movement $x^0(t)$ at $t\in[0,4]$ by coordinates $x_1^0(t)$, $x_2^0(t)$, $x_3^0(t)$, $x_4^0(t)$.

Note, by direct substitution, we can verify that the obtained optimal motions satisfy condition (1.4), i.e.,

$$F_1x^0(t_1)+F_2x^0(t_2)=\begin{pmatrix}3\\2\end{pmatrix}.$$

Thus, we have obtained explicit expressions for the optimal control and the corresponding optimal motion for system (1.2) with given initial and final values of the phase vector and nonseparated intermediate conditions (1.4).

V. Conclusion

The problem of optimal control of the motion of a two-link planar manipulator on a fixed base with given initial and final conditions and nonseparated conditions for the values of the phase vector at intermediate times is solved. The application of the proposed approach is exemplified by the construction of the functions of optimal control and the corresponding optimal motion with given nonseparated conditions for the values of the phase vector coordinates at some two intermediate times. The constructed corresponding graphs for the coordinates of the phase vector of the manipulator confirm the results obtained.

Acknowledgment

The research of S. Solodusha was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (Project FWEU-2021-0006, theme no. AAAA-A21-121012090034-3).

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